Teaching harmonic motion in trigonometry: Inductive inquiry supported by physics simulations

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In this paper, we would like to present a lesson whose goal is to utilise a scientific environment to immerse a trigonometry student in the process of mathematical modelling. The scientific environment utilised during this activity is a physics simulation called *Wave on a String* created by the PhET Interactive Simulations Project at Colorado University at Boulder and available free on the Internet. The outline of the activity, situated in inductive inquiry, is written in a format that is adaptable to various classroom settings; students can work independently in front of a computer or in groups. If a computer lab is not available, the simulation can be projected on a screen in a regular math classroom. In all of these settings, the teacher takes the role of a facilitator. Although, the lesson was developed following trigonometry curriculum in the US, its cognitive learning objectives fit well into the scope of the proposed Australian mathematics curriculum (ACARA, 2010) that also emphasises the development of the skills of mathematical modelling, data collection, and analysis. The activity, presenting applications of periodic functions in a non-geometric setting, can be conducted in Australian Upper Secondary or Lower Tertiary Trigonometry courses. With some extensions, including dumped oscillation, its content will fit into Queensland Mathematics C syllabus (QSA, 2008), in particular the section of Advanced Periodic and Exponential Functions.

Inductive reasoning

Inductive reasoning is a thought process whose ultimate goal is knowledge acquisition. Inductive reasoning encompasses a range of instructional methods, including inquiry learning, problem-based learning, project-based learning, case-based teaching, discovery learning, and just-in-time teaching (Prince & Felder, 2006). Inductive reasoning is commonly applied in science, where data is gathered and mathematical models are formulated to predict future behaviour of highlighted quantities. Literature (Felder & Brent 2004) shows that challenges provided by inductive methods serve as precursors to

students' intellectual development.

Learning methods involving inductive reasoning are characterised as constructivist methods. They build on the widely accepted principle that students construct their own versions of reality rather than simply absorb versions presented by their instructors. In *cognitive constructivism*, which originated primarily in the work of Piaget (1972) an individual's reactions to experiences lead to learning.

Although inductive reasoning produces multiple learning outcomes and it is extensively used in sciences (Thacker, Eunsook, Trefz, & Lea, 1994), this inquiry method is rarely used in trigonometry. We argue that applying trigonometric functions to model harmonic motion can provide a rich scientific context to exercise mathematical modelling through inductive inquiry in trigonometry classes as well.

Adopted learning method

Inquiry learning, one of the simplest forms of the inductive reasoning was selected to construct this activity. Staver and Bay (1987) identified three stages of inquiry learning: *structured inquiry*, where students are given a problem and an outline for how to solve it, *guided inquiry* where in addition, students are supposed to figure out the solution method, and *open inquiry* where students must formulate the problem and find the solution.

We situated this activity in a guided inquiry stage. By modifying some of its auxiliary elements, it can be changed to either structured or open inquiry depending on students' responses.

The framework of the guided inquiry follows Joice's (2009) four layers of inductively organised learning environment: *focus*, *conceptual control*, *inference*, and *confirmation*. In order to parallel this inquiry with its scientific counterpart, problem statement was included as a catalyst of the process. The following descriptions of the layers served as theoretical foundations of the activity. Problem statement is a form of a question that students answer because of conducting an experiment. Focus is building (collecting) data and asking students to analyse the attributes of the data and formulate the hypothesis. Conceptual control (analysis) is classifying the facts and identifying patterns of regularity. Inference is a generalisation (formulation of a pattern or law) about the relations between the collected facts that leads to acquiring a general formula or mathematical function. Confirmation is a verification of the derived model in new (physical) circumstances conducted through testing inference and further observations.

An effective implementation of the inquiry method provides also the students with practicing a conduct of scientific experiment of how to identify and collect appropriate evidence, analyse and interpret results, formulate conclusions, and evaluate the conclusions (Lee, 2004). Since understanding physics concepts requires formal reasoning, familiarity with the process of mathematisation of generated data, and extracting general principles from specific cases (Bellomonte, Guastella, & Sperandeo-Mineo, 2005), exposing

trigonometry students to a simulated physics phenomena might produce multiple results. It not only develops their modelling skills but also helps them understand the laws of physics. Furthermore, while conducting this activity, students will be placed in the roles of scientists actively constructing new knowledge. By referring to a scientific environment, practitioners will learn to select information based on scientific validity, a cognitive skill that they can apply in other subject areas as well as in their work places.

Why physics simulations

Physics simulations selected for this project are provided free online by PhET Interactive Simulations Project at Colorado University. Although their primary purpose is enhancing the teaching of physics, we argued that they could be integrated into the process of teaching and learning of mathematics. While working on the virtual labs, students can state hypotheses, observe scientific processes, take measurements, construct mathematical models, and validate them. They can modify the variables of the experiments, as well as predict and verify the respective outputs. Inquiry conducted by Lima (2010) shows that when prediction is used effectively students are likely to progress from passive listeners to active thinkers, simultaneously expanding and deepening their mathematical knowledge. With the aid of graphing technology, derived mathematical models can be further verified. Research conducted by PhET (Finkelstein, Perkins, Adams, Kohl, & Podolefsky, 2004) showed that these simulations can be substituted effectively for real laboratory equipment in physics courses. Findings of research (Sokolowski & Walters, 2010, p. 110) conducted in a South-Central Texas high school proved that mathematics students not only learned more and scored higher on the district and state standardised test items related to analysis and synthesis but that they also enjoyed and appreciated the new learning environment.

The structure of the activity

The activity uploaded also at http://phet.colorado.edu/en/contributions/view/3340 evolves within the five stages of guided inquiry. The purpose of the commentaries is to help walk students through the inquiry process.

Introduction of the concept and demonstration of the simulation

The teacher opens the simulation at http://phet.colorado.edu/en/simulation/wave-on-a-string and demonstrates its features, focusing the students' attention on the shape of the string while it transmits energy. The oscillations can be generated manually, or they can be produced periodically by checking the button **Oscillate** located on the left side of the simulation.

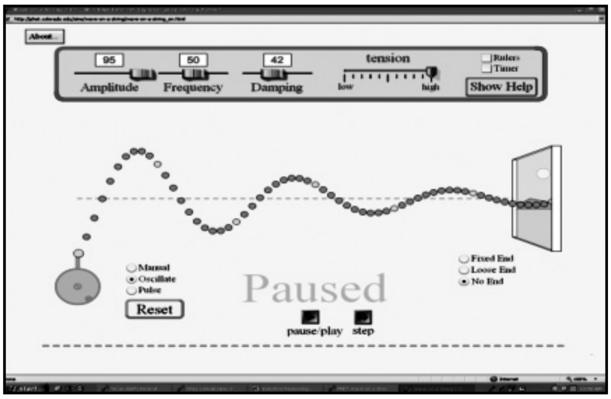

Figure 1. General demonstration of the features of the simulation. Source: PhET Interactive Simulations, University of Colorado, http://phet.colorado.edu.

The instructor might begin the process with generating the wave manually and mention to students some obstacles that they would face to mathematically describe this irregular movement. The oscillator, a wheel rotating periodically due assigned frequency, produces a regular wave on the string. It is important that the **No End** mode located on the bottom right corner of the simulation is checked. The instructor might also want to demonstrate the effect of the different damping factors and string tensions on the motion of the energy. At last, the instructor directs students' attention to frequency and amplitude, as these two physical factors of the periodic motion will dictate the formulation of respective periodic function. Since mathematical modelling refers to successive approximation, yet more regularity in the movement would help construct the model. This can be achieved by reducing the damping factor to zero. Under these circumstances, the energy transmitted by string is not dissipated to the environment, and therefore the amplitude of the wave indicating the amount of energy remains constant. It might be interesting to students to note that the damping factor does not affect the frequency of the wave. The frequency of the wave depends only on the frequency of the source producing the wave.

Problem statement

What mathematical function can be applied to model the path of motion of the energy generated by the wheel? What will the independent and dependent variables of the function be? Students might be given some time to discuss their answers in groups.

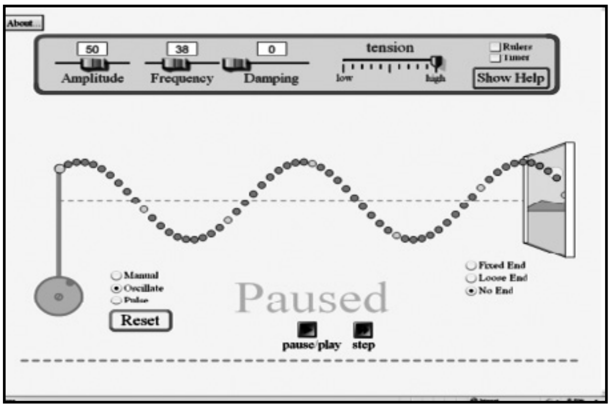

Figure 2. Form of the wave to discuss the problem statement. Source: PhET Interactive Simulations, University of Colorado, http://phet.colorado.edu.

The instructor elicits from the students a periodic nature of the motion that suggests sinusoidal function to be applied. There is one important element that the instructor might want to address to the students at this point. Since the path of the motion is two- dimensional (the energy oscillates up and down and moves forward) there can be two different ways undertaken to model the movement:

A. Applying parametric representation to model independently vertical $y = f(t)$ and horizontal position $x = f(t)$ of the front of the wave.
In this case, the vertical movement would be modelled by a sinusoidal function and horizontal by a linear (energy moves with a constant forward velocity). Both functions would be expressed in terms of time. Although, this model depicting a dynamics of the system is physically rich, students who did not study properties of parametric equations might find it difficult to apply. Therefore, another (B), a simplified version of this representation is suggested to be used.

B. Expressing vertical position of the wave in terms of horizontal position, $y = f(x)$
This representation is easily conveyable to an average trigonometry student and it is suggested to be adopted to model the sinusoidal path. If an opportunity exists, the instructor might want to prove mathematically that the parametric forms discussed in A can be converted into a singular representation discussed in B.

Focus/gathering information/stating hypothesis

In this part, the teacher discusses, in detail, the critical components of the sinusoidal function such as amplitude, period, horizontal, and vertical shift, and how these quantities can be identified and measured in the experiment.

Due to chosen singular model, the period of anticipated function will be expressed in the units of metres (here millimetres). In physics, the length of one wave, expressed in metres, represents the wavelength of the wave denoted by λ. Although students who took physics course will correlate the distance to wavelength, for the purpose of this activity, the distance will be labelled Δx.

The instructor might want to demonstrate some measuring devices embedded in the simulation, such as a ruler and a stopwatch that help quantify the highlighted quantities.

Establishing a frame of reference (x- and y-axes) is also important. Due to its location, function vertical and horizontal shifts will be referenced. It is suggested that to develop the general model, the x-axis is established at the equilibrium line of the string and the y-axis is aligned with the centre of the oscillator (initial position of the propagated energy).

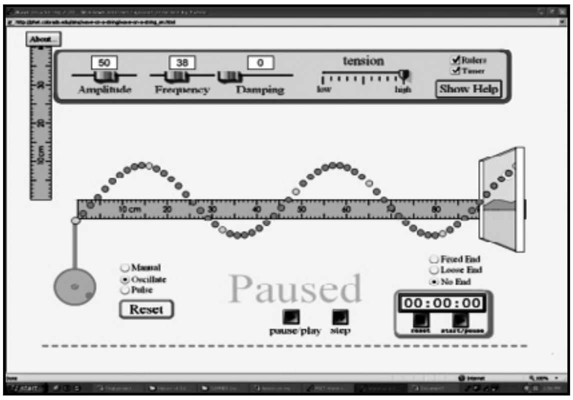

Figure 3. Preparing the simulation to gather necessary parameters. Source: PhET Interactive Simulations, University of Colorado, http://phet.colorado.edu.

Analysis/conceptual control

Once a tentative model has been concluded, students focus on measuring necessary quantities that will constitute the form of the sinusoidal function. In order to further generalise the model, only one full cycle can be shown on the screen for the analysis. Students can be given rulers and be asked to measure necessary quantities using screenshots of the simulation copied in their lab outlines. This approach has some advantages; it makes the activity more tangible. Students prefer this approach to just using the numbers embedded in the scenario. The lengths can be expressed in millimetres or centimetres.

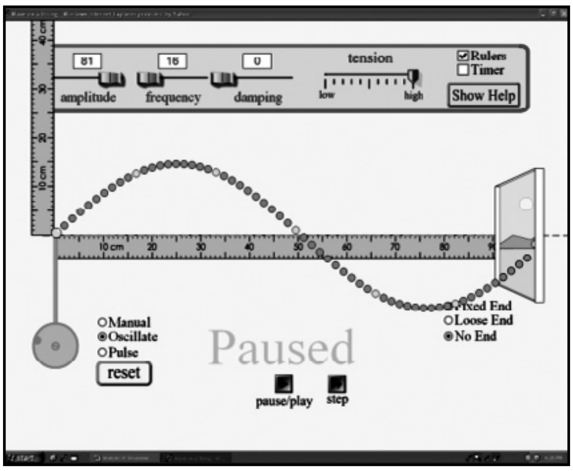

Figure 4. Quantifying parameters. Source: PhET Interactive Simulations, University of Colorado, http://phet.colorado.edu.

Generalisation of the analysis/inference

During this stage, students transfer measured quantities into a mathematical model. In order to review essential elements of a periodic function, they can work on multiple-choice questions. Samples of such items are provided below.

A general form of a periodic function is given by $y = A\sin\left(\frac{2\pi}{T}\right)t$.

1. Select the quantity that represents the measured average time of one full cycle.
   A. 2π B. Amplitude (A)
   C. Period (T) D. General time variable (t)
2. Select the quantity that represents the dependent variable in this function.
   A. Amplitude (A) B. Vertical position of wave (y)
   C. General time variable (t) D. Period (T)
3. Suppose that the length of one full cycle of a wave is denoted by Δx and expressed in the units of metres. Which expression can be used to model the wave?
   A. $y = A\sin\left(\frac{2\pi}{\Delta x}\right)x$ B. $y = A\sin\left(\frac{2\pi}{\Delta x}\right)t$
   C. $y = A\sin\left(\frac{2\pi}{T}\right)\Delta x$ D. $y = A\sin\left(\frac{2\pi}{x}\right)\Delta x$

Following the review, the teacher discusses the answers and then lets the students construct their functions. Students substitute the values of these quantified components to the general form of

$$y = A\sin\left(\frac{2\pi}{\Delta x}\right)x$$

and then verify the function equation. Building on Bateman (1990), inductive instruction should be spirally organised; students should be directed to continually revisit critical concepts and improve their cognitive models, thus verification, the next stage of the inquiry follows.

Verification and confirmation of the derived model

This stage is very significant in the process of the guided inquiry. When working on typical paper-and-pencil problems, this stage is often omitted because the physical representation of a wave is usually not provided and the movement is not observable. The availability of the simulation presents a great opportunity for contrasting observed wave with its mathematical representation. Students can use a graphing calculator or any technological tool (sketchpad) that converts algebraic function into a graph. They might be asked to determine the dimensions of the window of a graphing calculator so that their functions resemble the screen shot as closely as possible.

If graphing technology is not available, students can verify the model the old fashioned way using a table of values. They could calculate outputs for

selected inputs and compare the values with the model. Further verification can refer to modification of the components of the *parent* function. Sample questions in regards to this mode of verification are presented below. For each modification, the students are supposed to; use the simulation to observe the change, write new function, and use graphing technology to check if derived function corresponds to observable wave.

1. Suppose that the *x*-axis—the draggable reference (dotted) line shown on the simulation—is moved 30 mm below the string. Which component of the derived function should be changed to reflect this transformation?
   A. Wavelength B. Vertical transformation
   C. Period D. Horizontal transformation
2. Due to a modified frequency, there are twice as many waves observed on the string. Which component of the sinusoidal function should be changed to reflect this transformation?
   A. Amplitude B. Vertical transformation
   C. Period D. Horizontal compression
3. Suppose that the maximum height of the wave as measured from the equilibrium line increased by 10 cm. Which parameter changed?
   A. Amplitude B. Vertical transformation
   C. Period D. Horizontal compression

Exchanging thoughts about the experiment, and suggesting ways of improving this learning environment concludes the activity.

What impact do the simulations have on student learning process?

Students highly praise the new learning environment and find the lessons utilising simulations very attractive. As we mentioned before, this environment affects positively their test scores. Following this positive feedback, more simulations were adopted to enhance the modelling process in other sections of mathematics such as polynomial or transcendental functions. The simulations were also utilised in calculus to enhance the teaching of limits, derivatives, and integrals including the First Fundamental Theorem of Calculus. We argue that the *virtual* physical world and the inquiry processes that the students were immersed in to model the world helps them to understand the modelling process and prepares them for engineering classes and college. We foresee a need for a more systematic research study of the influence of the simulations on mathematical knowledge acquisition by mathematics students. We hope that the readers will find the journey interesting enough to also get involved in similar research.

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